Division of Polynomials

Exercises

1. Divide the polynomial $2x^3 - 7x^2 - 4x + 36$ by each of the following divisors. Express the polynomial in terms of the quotient, divisor, and remainder. State any restrictions on the variable.
   a. $x - 1$
   b. $x + 3$
   c. $2x - 3$
2. Express each division in terms of the quotient, divisor, and remainder.
   a. $\dfrac{x^3 + x - 8x^2 + 37}{x + 4}, x \neq -4$
   b. $\dfrac{10x^3 - 9x^2 - 8x + 11}{5x - 2}, x \neq \frac{2}{5}$
   c. $\dfrac{3x + 2 - x^3}{x - 2}, x \neq 2$
   d. $\dfrac{x^4 - 2x^2 + 12}{x - 4}, x \neq 4$
   e. $\dfrac{x^5 - 2x^4 - 7x^3 + 13x^2 + 2x - 18}{x^2 - 2x - 3}, x \neq -1, 3$
3. The polynomial $6x^3 - 5x^2 - 49x + 60$ is divided by $2x - 5$.
   a. Identify the restrictions on x.
   b. Show that the remainder is zero.
   c. Express the polynomial dividend in terms of the divisor, quotient, and remainder.
   d. What conclusion can be drawn when the remainder is zero?
   e. Express the polynomial in fully factored form.
4. a. Divide $f(x) = x^3 + (a + b)x^2 + (ab + c)x + ac$ by $d(x) = x + a$ and express $f(x)$ in terms of the divisor, quotient, and remainder.
   b. Using your findings in part a, create a cubic polynomial that has $x - 2$ as a factor. Verify your answer by carrying out the division.
5. a. When dividing a 5^{th} degree polynomial by a 2^{nd} degree polynomial, what is the degree of the quotient and the maximum degree of the remainder?
   b. When dividing a n^{th} degree polynomial by a divisor of degree m, where m and n are positive integers and $m \geq n$, what is the degree of the quotient and the maximum degree of the remainder?
6. When a polynomial $P(x)$ is divided by $x + 3$, the quotient is $3x^2 - 5x + 4$ and the remainder is -10. Find $P(x)$ in standard form.
7. Find the divisor given the divident, quotient, and remainder.
   a. The dividend is $3x^3 - 5x^2 - 7x - 1$, the quotient is $3x^2 + 4x + 5$, and the remainder is 14.
   b. The dividend is $2x^4 + 11x^3 + 5x^2 - 31x + 7$, the quotient is $2x^2 + 3x - 5$, and the remainder is $-8x + 2$.
8. The volume of a cylinder is given by $(\pi x^3 + 4\pi x^2 - 3\pi x - 18\pi)\ \text{cm}^3$. If the radius of the cylinder is $(x + 3)$ cm, determine the height of the cylinder in terms of x.
9. Given that
   $$\frac{x^4 - 3x^3 + px^2 - 11x - 7}{x^2 + 2x + 1} = x^2 + qx - 3 + \frac{r}{x^2 + 2x + 1} \text{ where } p, q, r \in \mathbb{R},$$
   find p, q, and r.
10. a. When a number n is divided by 7, the remainder is 3. What is the remainder when $4n$ is divided by 7?
    b. When $P(x)$ is divided by $(x + 1)$, the remainder is 3. What is the remainder when $xP(x)$ is divided by $(x + 1)$?
11. Prove that $\underbrace{2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{n-2} + 2^{n-1}}_{n \text{ terms}} = 2^n - 1$, by dividing $x^n - 1$ by $x - 1$.

Division of Polynomials

Partial Solutions

1. There is no solution provided for this question.

2. a. Ordering the terms of the dividend, $P(x) = x^3 - 8x^2 + x + 37$.
   Performing long division,

$$
\begin{array}{r|rrrrrrr}
 & & x^2 & - & 12x & + & 49 \\
\hline
x+4 & x^3 & - & 8x^2 & + & x & + & 37 \\
- & (x^3 & + & 4x^2) & & & & \\
\hline
 & & & -12x^2 & + & x & & \\
 & & - & (-12x^2 & - & 48x) & & \\
\hline
 & & & & & 49x & + & 37 \\
 & & & & - & (49x & + & 196) \\
\hline
 & & & & & & & -159
\end{array}
$$

Alternatively, by performing synthetic division (since the divisor is of the form $x - n$),

$$
\begin{array}{r|rrrr}
-4 & 1 & -8 & 1 & 37 \\
 & & -4 & 48 & -196 \\
\hline
 & 1 & -12 & 49 & -159
\end{array}
$$

Hence the quotient is $x^2 - 12x + 49$, the remainder is -159, and

$$\frac{x^3 + x - 8x^2 + 37}{x + 4} = x^2 - 12x + 49 + \frac{-159}{x+4}, x \neq -4$$

b. Performing long division,

$$
\begin{array}{r|rrrrrrr}
 & & & 2x^2 & - & x & - & 2 \\
\hline
5x-2 & 10x^3 & - & 9x^2 & - & 8x & + & 11 \\
- & (10x^3 & - & 4x^2) & & & & \\
\hline
 & & & -5x^2 & - & 8x & & \\
 & & - & (-5x^2 & + & 2x) & & \\
\hline
 & & & & & -10x & + & 11 \\
 & & & & - & (-10x & + & 4) \\
\hline
 & & & & & & & 7
\end{array}
$$

Alternatively, if we use synthetic division, the divisor must be of the form $x - n$. Note that the divisor $5x - 2$ can be expressed as $5(x - \frac{2}{5})$. Performing synthetic division with $n = \frac{2}{5}$,

$$
\begin{array}{r|rrrr}
\frac{2}{5} & 10 & -9 & -8 & 11 \\
 & & 4 & -2 & -4 \\
\hline
 & 10 & -5 & -10 & 7
\end{array}
$$

Using the division statement,

$$
\begin{aligned}
\frac{10x^3 - 9x^2 - 8x + 11}{5x - 2} &= \frac{1}{5}\left(\frac{10x^3 - 9x^2 - 8x + 11}{x - \frac{2}{5}}\right) \\
&= \frac{1}{5}\left(10x^2 - 5x - 10 + \frac{7}{x - \frac{2}{5}}\right) \qquad \text{from synthetic division above} \\
&= 2x^2 - x - 2 + \frac{7}{5x - 2}
\end{aligned}
$$

In either method, the quotient is $2x^2 - x - 2$, the remainder is 7, and

$$\frac{10x^3 - 9x^2 - 8x + 11}{5x - 2} = 2x^2 - x - 2 + \frac{7}{5x - 2}, x \neq \frac{5}{2}$$

c. Ordering terms and assigning placeholders, $P(x) = -x^3 + 0x^2 + 3x + 2$. Performing long division,

$$
\begin{array}{r|rrrrrrr}
 & & & -x^2 & - & 2x & - & 1 \\
\hline
x-2 & -x^3 & + & 0x^2 & + & 3x & + & 2 \\
- & (-x^3 & + & 2x^2) & & & & \\
\hline
 & & & -2x^2 & + & 3x & & \\
 & & - & (-2x^2 & + & 4x) & & \\
\hline
 & & & & & -x & + & 2 \\
 & & & & - & (-x & + & 2) \\
\hline
 & & & & & & & 0
\end{array}
$$

Thus the quotient is $-x^2 - 2x - 1$, the remainder is 0, and

$$\frac{3x + 2 - x^3}{x - 2} = -x^2 - 2x - 1, x \neq 2$$

d. Assigning placeholders, $P(x) = x^4 + 0x^3 - 2x^2 + 0x + 12$.

Performing long division,

$$\begin{array}{r|l}
 & x^3 + 4x^2 + 14x + 56 \\
\hline
x - 4 & x^4 + 0x^3 - 2x^2 + 0x + 12 \\
 & -(x^4 - 4x^3) \\
 & \quad 4x^3 - 2x^2 \\
 & \quad -(4x^3 - 16x^2) \\
 & \qquad 14x^2 + 0x \\
 & \qquad -(14x^2 - 56x) \\
 & \qquad\quad 56x + 12 \\
 & \qquad\quad -(56x - 224) \\
 & \qquad\qquad 236
\end{array}$$

Alternatively, by performing synthetic division (since the divisor is of the form $x - n$),

$$\begin{array}{r|rrrrr}
4 & 1 & 0 & 2 & 0 & 12 \\
 & & 4 & 16 & 56 & 224 \\
\hline
 & 1 & 4 & 14 & 56 & 236
\end{array}$$

Therefore the quotient is $x^3 + 4x^2 + 14x + 56$, the remainder is 236, and

$$\frac{x^4 - 2x^2 + 12}{x - 4} = x^3 + 4x^2 + 14x + 56 + \frac{236}{x - 4}, x \neq 4$$

e. Performing long division,

$$\begin{array}{r|l}
 & x^3 + 0x^2 - 4x + 5 \\
\hline
x^2 - 2x - 3 & x^5 - 2x^4 - 7x^3 + 13x^2 + 2x - 18 \\
 & -(x^5 - 2x^4 - 3x^3) \\
 & \quad -4x^3 + 13x^2 + 2x \\
 & \quad -(-4x^3 + 8x^2 + 12x) \\
 & \qquad 5x^2 - 10x - 18 \\
 & \qquad -(5x^2 - 10x - 15) \\
 & \qquad\quad -3
\end{array}$$

Therefore the quotient is $x^3 - 4x + 5$, the remainder is -3, and

$$\frac{x^5 - 2x^4 - 7x^3 + 13x^2 + 2x - 18}{x^2 - 2x - 3} = x^3 - 4x + 5 + \frac{-3}{x^2 - 2x - 3}, x \neq -1, 3$$

3. There is no solution provided for this question.

4. a.

$$\begin{array}{r|l}
 & x^2 + bx + c \\
\hline
x + a & x^3 + (a + b)x^2 + (ab + c)x + ac \\
 & -(x^3 + ax^2) \\
 & \quad bx^2 + (ab + c)x \\
 & \quad -(bx^2 + abx) \\
 & \qquad cx + ac \\
 & \qquad -(cx + ac) \\
 & \qquad\quad 0
\end{array}$$

Alternatively, by performing synthetic division with $n = -a$,

$$\begin{array}{r|cccc}
-a & 1 & a + b & ab + c & ac \\
 & & -a & -ab & -ac \\
\hline
 & 1 & b & c & 0
\end{array}$$

Thus the quotient is $x^2 + bx + c$, the remainder is 0, and

$$f(x) = (x + a)(x^2 + bx + c)$$

b. From part a), answers will be of the form $f(x) = x^3 + (a + b)x^2 + (ab + c)x + ac$ where $d(x) = x + a$ is a factor. If the factor is $x - 2$, then $a = -2$; choosing $b = 1$ and $c = 3$, $f(x) = x^3 - x^2 + x - 6$.

Verifying,

$$\begin{array}{r|rrrr}
2 & 1 & -1 & 1 & -6 \\
 & & 2 & 2 & 6 \\
\hline
\end{array}$$

1	1	3	0

The remainder is 0 so $f(x) = x^3 - x^2 + x - 6 = (x-2)(x^2 + x + 3)$ has a factor of $x - 2$.

5. There is no solution provided for this question.

6. The general expression for the dividend $P(x)$ in terms of the quotient, $q(x)$, the divisor $d(x)$, and the remainder $r(x)$ is

$$P(x) = q(x)d(x) + r(x)$$

Thus,

$$\begin{aligned} P(x) &= (3x^2 - 5x + 4)(x+3) - 10 \\ &= 3x^3 - 5x^2 + 4x + 9x^2 - 15x + 12 - 10 \\ &= 3x^3 + 4x^2 - 11x + 2 \end{aligned}$$

7. There is no solution provided for this question.

8. The formula for the volume of a cylinder is $V = \pi r^2 h$ where r is the radius and h is the height.

The radius is given as $x + 3$, and $V = \pi x^3 + 4\pi x^2 - 3\pi x - 18\pi$. Substituting the expression for r into $V = \pi r^2 h$:

$$\begin{aligned} \pi(x+3)^2 h &= \pi x^3 + 4\pi x^2 - 3\pi x - 18\pi \\ (x+3)^2 h &= x^3 + 4x^2 - 3x - 18 \\ h &= \frac{x^3 + 4x^2 - 3x - 18}{(x+3)^2} \\ h &= \frac{x^3 + 4x^2 - 3x - 18}{x^2 + 6x + 9} \end{aligned}$$

Dividing,

						x		$-$	2
$x^2 + 6x + 9$		x^3	$+$	$4x^2$	$-$	$3x$		$-$	18
	$-$	$(x^3$	$+$	$6x^2$	$+$	$9x)$			
				$-2x^2$	$-$	$12x$		$-$	18
			$-$	$(-2x^2$	$-$	$12x$		$-$	$18)$
									0

Hence the height is $(x - 2)$ cm.

9. There is no solution provided for this question.

10. a. The division statement for integers is

$$\frac{n}{d} = q + \frac{r}{d}$$

where n is the integer divident, d is the divisor, q is the quotient, and r is the remainder such that $0 \le r < d$ and $d, q, r \in \mathbb{Z}$.
From the question,

$$\frac{n}{7} = q + \frac{3}{7}$$

where q is the quotient of the division, $d = 7$, and $r = 3$. Then

$$\begin{aligned} \frac{4n}{7} &= 4q + \frac{12}{7} \\ \frac{4n}{7} &= 4q + 1 + \frac{5}{7} && \text{so that } r < d \\ \frac{4n}{7} &= (4q + 1) + \frac{5}{7} \\ \frac{4n}{7} &= u + \frac{5}{7} \end{aligned}$$

where $u = 4q + 1$ is the quotient, $u \in \mathbb{Z}$. Therefore the remainder is 5.

b. We can express the statement mathematically as

$$\frac{P(x)}{x+1} = q(x) + \frac{3}{x+1}$$

where $q(x)$ is the quotient of the division, of degree one less than $P(x)$. Then

$$\begin{aligned} \frac{xP(x)}{x+1} &= xq(x) + \frac{(3x+3) - 3}{x+1} \\ &= xq(x) + 3 + \frac{-3}{x+1} \\ &= (xq(x) + 3) + \frac{-3}{x+1} \\ &= u(x) + \frac{-3}{x+1} \end{aligned}$$

where $u(x) = xq(x) + 3$ is the quotient, of degree one less than $xP(x)$ since $q(x)$ is of degree one less than $P(x)$. Therefore the remainder of the division is -3.

11. Let $f(x) = x^n - 1$. Divide $f(x)$ by $(x - 1)$ using long division gives

			x^{n-1}	$+$	x^{n-2}	$+$	x^{n-3}	$+$	$\cdots$	$+$	x	$+$	1		
$x - 1$	x^n	$+$	0x^{n - 1}	$+$	0x^{n - 2}	$+$	0x^{n - 3}	$+$	\cdots	$+$	0x	-	1		

$$-(x^n - x^{n-1})$$

$$x^{n-1} + 0x^{n-2}$$

$$-(x^{n-1} - x^{n-2})$$

$$x^{n-2} + 0x^{n-3}$$

$$-(x^{n-2} - x^{n-3})$$

$$x^{n-3} + \cdots$$

$$\ddots$$

$$x - 1$$

$$-(x - 1)$$

$$0$$

Therefore, $f(x) = x^n - 1 = (x - 1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$ Substitute $x = 2$ to obtain

$$\begin{align*} f(2) &= (2 - 1)(2^{n-1} + 2^{n-2} + \cdots + 2 + 1) \\ 2^n - 1 &= (1)(2^{n-1} + 2^{n-2} + \cdots + 2 + 1) \\ 2^n - 1 &= 2^{n-1} + 2^{n-2} + \cdots + 2 + 2^0 \end{align*}$$